

SCHWARZ LEMMA FOR CONICAL KÄHLER METRICS WITH GENERAL CONE ANGLES

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ABSTRACT. The Schwarz–Pick lemma is a fundamental result in complex analysis. It is well-known that Yau generalized it to the higher dimensional manifolds by applying his maximum principle for complete Riemannian manifolds. Jeffres obtained Schwarz lemma for volume forms of conical Kähler metrics, based on a barrier function and the maximum principle argument. In this note, we generalize Jeffres’ result to general cone angles including the case when the pullback of the metric would blows up along the divisors.

1. INTRODUCTION

The Schwarz–Pick lemma states that any holomorphic map between the unit disks in the complex plane decreases the Poincaré metrics. After that, Ahlfors [Ahl38] generalized it to a holomorphic map from the unit disk to a hyperbolic Riemann surface. For higher dimensions, Yau [Yau78] showed that any holomorphic map from complete Kähler manifold whose Ricci curvature is bounded from below to a Hermitian manifold whose holomorphic bisectional curvature is bounded by a negative constant decreases the metric up to a multiplicative constant. Also, he showed that, under similar conditions on curvatures, any holomorphic map decreases the volume forms up to a multiplicative constant. Both results essentially based on his maximum principle for complete Riemannian manifolds. Later on, many generalizations obtained in various geometric settings.

In this note, we focus on the conical Kähler metrics, for short, cone metrics. Let X be a compact Kähler manifold of dimension n , D be a smooth divisor on X , and β be a real number satisfying $0 < \beta < 1$. The cone metric ω with cone angle $2\pi\beta$ along D is a Kähler metric on $X \setminus D$ which is locally quasi-isometric to the standard cone metric

$$\omega_\beta := \frac{\beta^2}{|z|^{2(1-\beta)}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 + \sum_{i=2}^n \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i,$$

and satisfies some regularity conditions. (For a precise definition of the cone metric, see Definition 2.2.) The notion of cone metrics plays an important role in recent advances in Kähler geometries, in particular Kähler–Einstein problems, for instance see [CDS15a, CDS15b, CDS15c, Tia15].

To state the theorems, we use the following setups and notations.

Setup 1.1. Let X and Y be compact Kähler manifolds, $D \subset X$, $E \subset Y$ be smooth divisors, and $f: X \rightarrow Y$ be a surjective holomorphic map satisfying

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$f^*(E) = kD$ with $k \in \mathbb{Z}_{>0}$. Let ω_X (resp. ω_Y) be a cone metric with cone angle $2\pi\alpha$ (resp. $2\pi\beta$) along D (resp. E) on X (resp. Y). Let $s \in H^0(X, \mathcal{O}_X(D))$ be a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D and h be a smooth Hermitian metric on it satisfying $|s|_h \leq 1$. Let $C > 0$ be an upper bound for the Chern curvature of h i.e. $\sqrt{-1}R_h \leq C\omega_X$. For a Kähler form ω , we will denote by $\text{Ric}(\omega)$ the Ricci curvature of ω , $R(\omega)$ the scalar curvature of ω , and $\text{Bisect}(\omega)$ the bisectional curvature of ω .

Schwarz lemma for the cone metrics obtained by Jeffres [Jef00a] is states as follows.

Theorem 1.2 ([Jef00a, Theorem]). *Assume that $\dim X = \dim Y = n$, the cone angles satisfy $\alpha \leq \beta$ and there exists non-negative constants $A, B \geq 0$ satisfying*

$$(1.3) \quad R(\omega_X) \geq -A, \quad \text{Ric}(\omega_Y) \leq -B\omega_Y < 0.$$

Then, the volume forms satisfy

$$f^*\omega_Y^n \leq \left(\frac{A}{B}\right)^n \omega_X^n \quad \text{on } X \setminus D.$$

Since the cone metric is not complete on $X \setminus D$, we cannot apply the maximum principle argument directly. Jeffers overcame this difficulty by using a barrier function, called “Jeffres’ trick”. However, his original proof seems to need more assumptions on the regularity of the cone metrics along D as in Definition 2.1 (see the proof of Proposition 3.3).

In this note, we will generalize this theorem to a general cone angle and prove a Schwarz lemma for cone metrics.

Theorem 1.4 (Volume forms). *Assume that $\dim X = \dim Y = n$ and the curvature condition (1.3) holds.*

(a) *Suppose $\alpha \leq k\beta$. Then we have*

$$f^*\omega_Y^n \leq \left(\frac{A}{nB}\right)^n \omega_X^n \quad \text{on } X \setminus D.$$

(b) *Suppose $\alpha > k\beta$. Then we have*

$$f^*\omega_Y^n \leq \left(\frac{A + (\alpha - k\beta)C}{nB}\right)^n \frac{\omega_X^n}{|s|_h^{2(\alpha - k\beta)}} \quad \text{on } X \setminus D.$$

We remark that the condition $\alpha \leq k\beta$ on cone angles in the statement (a) is weaker than assumptions in Theorem 1.2.

Theorem 1.5 (Metrics). *Assume that there exists non-negative constants $A, B \geq 0$ such that the curvatures satisfy the following:*

$$(1.6) \quad \text{Ric}(\omega_X) \geq -A\omega_X, \quad \text{Bisect}(\omega_Y) \leq -B < 0.$$

(a) *Suppose $\alpha \leq k\beta$. Then we have*

$$f^*\omega_Y \leq \frac{A}{B}\omega_X \quad \text{on } X \setminus D.$$

(b) Suppose $\alpha > k\beta$. Then we have

$$f^*\omega_Y \leq \frac{A + (\alpha - k\beta)C}{B} \frac{\omega_X}{|s|_h^{2(\alpha - k\beta)}} \quad \text{on } X \setminus D.$$

If the cone angle satisfies $\alpha > k\beta$, the pullback $f^*\omega_Y$ has singularities along D . In fact, even in a one-dimensional case, the pullback of the standard cone metric $\omega_\beta = (\beta^2/|w|^{2(1-\beta)})\sqrt{-1}dw \wedge d\bar{w}/2$ by $f : z \mapsto w = z^k$ is given by

$$f^*\omega_\beta = \beta^2 k^2 |z|^{2(k\beta-1)} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z},$$

therefore we have

$$\frac{f^*\omega_\beta}{\omega_\alpha} = \frac{\beta^2 k^2}{\alpha^2} |z|^{2(k\beta-\alpha)},$$

which is singular if $\alpha > k\beta$.

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2. CONE METRICS

In this section, we recall the definition of cone metrics following [Don12, Section 4]. Let X be a compact Kähler manifold of dimension n , D be a smooth divisor on X , and β be a real number satisfying $0 < \beta < 1$. We first remark that if we take a local holomorphic chart $(U, (z^1, \dots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$, the standard cone metric ω_β induces a distance function d_β on U which is expressed as

$$d_\beta(z, w) = \left(|(z^1)^\beta - (w^1)^\beta|^2 + |z^2 - w^2|^2 + \dots + |z^n - w^n|^2 \right)^{1/2},$$

where $z = (z^1, \dots, z^n)$, $w = (w^1, \dots, w^n)$. Here, we take a suitable branch of z^β .

Definition 2.1 ($C^{2,\alpha,\beta}$ -functions). Let α be a constant satisfying $0 < \alpha < \min\{1/\beta - 1, 1\}$. We define the regularities of functions along D as follows.

- (1) A function f on X is said to be of class $C^{\alpha,\beta}$ if for any local holomorphic chart $(U, (z^1, \dots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$, f is an α -Hölder continuous function on U with respect to the distance function d_β .

This definition is equivalent to the following statement which is the original definition in [Don12]. We set \tilde{f} by $\tilde{f}(\xi, z^2, \dots, z^n) := f(|\xi|^{1/\beta-1} \xi, z^2, \dots, z^n)$. Then \tilde{f} is an α -Hölder continuous function with respect to ξ, z^2, \dots, z^n with respect to the Euclidean distance.

- (2) A $(1, 0)$ -form τ is said to be of class $C^{\alpha,\beta}$ if

$$\begin{aligned} |z^1|^{1-\beta} \tau \left(\frac{\partial}{\partial z^1} \right) &\in C^{\alpha,\beta}, \\ \tau \left(\frac{\partial}{\partial z^i} \right) &\in C^{\alpha,\beta} \quad \text{for } i = 2, \dots, n \end{aligned}$$

(3) A $(1, 1)$ -form σ is said to be of class $C^{\alpha, \beta}$ if

$$\begin{aligned} |z^1|^{2(1-\beta)} \sigma \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \bar{z}^1} \right) &\in C^{\alpha, \beta}, \\ |z^1|^{1-\beta} \sigma \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \bar{z}^i} \right) &\in C^{\alpha, \beta} \quad \text{for } i = 2, \dots, n, \\ |z^1|^{1-\beta} \sigma \left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^1} \right) &\in C^{\alpha, \beta} \quad \text{for } i = 2, \dots, n, \\ \sigma \left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j} \right) &\in C^{\alpha, \beta} \quad \text{for } i, j = 2, \dots, n. \end{aligned}$$

(4) A function f is said to be of class $C^{2, \alpha, \beta}$ if $f, \partial f, \bar{\partial} f, \sqrt{-1} \partial \bar{\partial} f$ are of class $C^{\alpha, \beta}$.

Definition 2.2 (Cone metrics). A closed positive $(1, 1)$ -current ω on X is called a *cone metric with cone angle $2\pi\beta$ along D* if it satisfies the following three conditions:

- (i) ω is a Kähler metric on $X \setminus D$
- (ii) For each point $x \in D$, there exists a local holomorphic chart $(U, (z^1, \dots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$ such that ω is quasi-isometric to ω_β on $U \setminus D$, that is, there exists a constant $C = C_U > 0$ such that

$$\frac{1}{C} \omega_\beta \leq \omega \leq C \omega_\beta \quad \text{on } U \setminus D.$$

Here, ω_β is the standard cone metric defined by

$$\omega_\beta := \frac{\beta^2}{|z|^{2(1-\beta)}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\bar{z}^1 + \sum_{i=2}^n \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i.$$

- (iii) There exists a smooth Kähler form ω_0 on X , and a $C^{2, \alpha, \beta}$ -function φ such that

$$\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi.$$

In [Jef00a], the regularity condition (iii) does not assumed. However, we assume here.

A typical example of the cone metric is $\omega := \omega_0 + \delta \sqrt{-1} \partial \bar{\partial} |s|_h^\beta$, where ω_0 is a smooth Kähler metric on X , δ is a sufficiently small constant, $s \in H^0(X, \mathcal{O}_X(D))$ is a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D , and h is a smooth Hermitian metric.

3. PROOF OF THE THEOREMS

To prove the theorem, we need the following Laplacian estimates which are obtained by [Che68, Lu68]. For the readers convenience, we prove here.

Proposition 3.1. *Let X, Y be (not necessarily compact) Kähler manifolds, and $f: X \rightarrow Y$ be a holomorphic map. Let ω_X (resp. ω_Y) be a smooth Kähler metric on X (resp. Y). We set $v := f^* \omega_Y^n / \omega_X^n$, and $u := \text{tr}_{\omega_X} (f^* \omega_Y)$.*

(a) Suppose that there exists non-negative constants $A, B \geq 0$ satisfying $R(\omega_X) \geq -A$, $\text{Ric}(\omega_Y) \leq -B\omega_Y$, and $\dim X = \dim Y = n$. Then we have

$$\begin{aligned}\Delta_{\omega_X} \log v &\geq nBv^{1/n} - A, \\ \Delta_{\omega_X} v &\geq v(nBv^{1/n} - A).\end{aligned}$$

(b) Suppose that there exists non-negative constants $A, B \geq 0$ satisfying $\text{Ric}(\omega_X) \geq -A\omega_X$, $\text{Bisect}(\omega_Y) \leq -B\omega_Y$. Then we have

$$\begin{aligned}\Delta_{\omega_X} \log u &\geq Bu - A, \\ \Delta_{\omega_X} u &\geq u(Bu - A).\end{aligned}$$

Proof. Let (z^1, \dots, z^n) and (w^1, \dots, w^n) be normal coordinates on X and Y respectively. We set

$$\omega_X = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j, \quad \omega_Y = \sqrt{-1}h_{\alpha\bar{\beta}}dw^\alpha \wedge d\bar{w}^\beta.$$

(a) v is locally denoted as

$$(3.2) \quad v = \frac{f^*\omega_Y^n}{\omega_X^n} = \frac{\det(h_{\alpha\bar{\beta}} \circ f) |\det J(f)|^2}{\det(g_{i\bar{j}})}$$

where $J(f)$ is the Jacobian of f . Therefore, on $\Omega := \{x \in X \mid \det J(f)(x) \neq 0\}$, we obtain

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial} \log v &= f^*\sqrt{-1}\partial\bar{\partial} \log \det(h_{\alpha\bar{\beta}}) + \sqrt{-1}\partial\bar{\partial} \log \det(g_{i\bar{j}}) - \sqrt{-1}\partial\bar{\partial} \log |\det J(f)|^2 \\ &= f^*(-\text{Ric}(\omega_Y)) + \text{Ric}(\omega_X).\end{aligned}$$

By the assumption on curvatures and the inequality of arithmetic and geometric means, we have the following estimates on Ω :

$$\begin{aligned}\Delta_{\omega_X} \log v &= \text{tr}_{\omega_X} (\sqrt{-1}\partial\bar{\partial} \log v) = \text{tr}_{\omega_X} (f^*(-\text{Ric}(\omega_Y))) + R(\omega_X) \\ &\geq B\text{tr}_{\omega_X} (f^*\omega_Y) - A \\ &\geq nBv^{1/n} - A, \\ \Delta_{\omega_X} v &= \Delta_{\omega_X} e^{\log v} = e^{\log v} (|\nabla \log v|_{\omega_X}^2 + \Delta_{\omega_X} \log v) \\ &\geq v\Delta_{\omega_X} \log v \\ &\geq v(nBv^{1/n} - A).\end{aligned}$$

By continuity, the last inequality holds on the whole X .

(b) We set

$$f^*\omega_Y = \sqrt{-1}h_{i\bar{j}}^*dz^i \wedge d\bar{z}^j := \sqrt{-1}(h_{\alpha\bar{\beta}} \circ f)(\partial_i f^\alpha)(\bar{\partial}_j \bar{f}^\beta)dz^i \wedge d\bar{z}^j,$$

and denote $R_{i\bar{j}k\bar{l}}$ and $S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ by the curvature tensor of ω_X and ω_Y respectively. Then we have the following inequalities, which are our assertion (b).

$$\begin{aligned}\Delta_{\omega_X} \operatorname{tr}_{\omega_X} (f^* \omega_Y) &= g^{k\bar{l}} \partial_k \partial_{\bar{l}} \left(g^{i\bar{j}} h_{i\bar{j}}^* \right) = g^{k\bar{l}} \left(\partial_k \partial_{\bar{l}} g^{i\bar{j}} \right) h_{i\bar{j}}^* + g^{k\bar{l}} g^{i\bar{j}} \left(\partial_k \partial_{\bar{l}} h_{i\bar{j}}^* \right) \\ &= g^{k\bar{l}} R_{k\bar{l}}^{i\bar{j}} h_{i\bar{j}}^* + \left(g^{k\bar{l}} g^{i\bar{j}} (\partial_i \partial_k f^\alpha) \overline{(\partial_{\bar{j}} \partial_{\bar{l}} f^\beta)} - g^{k\bar{l}} g^{i\bar{j}} (\partial_i f^\alpha) \overline{(\partial_{\bar{j}} f^\beta)} (\partial_k f^\gamma) \overline{(\partial_{\bar{l}} f^\delta)} S_{\alpha\bar{\beta}\gamma\bar{\delta}} \right) \\ &\geq \langle \operatorname{Ric}(\omega), f^* \omega_Y \rangle_{\omega_X} - (\partial_i f^\alpha) \overline{(\partial_{\bar{i}} f^\beta)} (\partial_k f^\gamma) \overline{(\partial_{\bar{k}} f^\delta)} S_{\alpha\bar{\beta}\gamma\bar{\delta}} \\ &\geq -Au + Bu^2 = u(Bu - A).\end{aligned}$$

$$\begin{aligned}\Delta_{\omega_X} \log \operatorname{tr}_{\omega_X} (f^* \omega_Y) &= \frac{\Delta_{\omega_X} \operatorname{tr}_{\omega_X} (f^* \omega_Y)}{\operatorname{tr}_{\omega_X} (f^* \omega_Y)} - \frac{|\nabla \operatorname{tr}_{\omega_X} (f^* \omega_Y)|_{\omega_X}^2}{(\operatorname{tr}_{\omega_X} (f^* \omega_Y))^2} \\ &= \frac{1}{\operatorname{tr}_{\omega_X} (f^* \omega_Y)} \left(\langle \operatorname{Ric}(\omega_X), f^* \omega_Y \rangle_{\omega_X} - g^{k\bar{l}} g^{i\bar{j}} (\partial_i f^\alpha) \overline{(\partial_{\bar{j}} f^\beta)} (\partial_k f^\gamma) \overline{(\partial_{\bar{l}} f^\delta)} S_{\alpha\bar{\beta}\gamma\bar{\delta}} \right. \\ &\quad \left. + g^{k\bar{l}} g^{i\bar{j}} (\partial_i \partial_k f^\alpha) \overline{(\partial_{\bar{j}} \partial_{\bar{l}} f^\beta)} \right) - \frac{|\nabla \operatorname{tr}_{\omega_X} (f^* \omega_Y)|_{\omega_X}^2}{(\operatorname{tr}_{\omega_X} (f^* \omega_Y))^2} \\ &= \frac{1}{\operatorname{tr}_{\omega_X} (f^* \omega_Y)} \left(\langle \operatorname{Ric}(\omega_X), f^* \omega_Y \rangle_{\omega_X} - g^{k\bar{l}} g^{i\bar{j}} (\partial_i f^\alpha) \overline{(\partial_{\bar{j}} f^\beta)} (\partial_k f^\gamma) \overline{(\partial_{\bar{l}} f^\delta)} S_{\alpha\bar{\beta}\gamma\bar{\delta}} \right) \\ &\quad + \frac{1}{(\operatorname{tr}_{\omega_X} (f^* \omega_Y))^2} \left(\operatorname{tr}_{\omega_X} (f^* \omega_Y) g^{k\bar{l}} g^{i\bar{j}} (\partial_i \partial_k f^\alpha) \overline{(\partial_{\bar{j}} \partial_{\bar{l}} f^\beta)} - |\nabla \operatorname{tr}_{\omega_X} (f^* \omega_Y)|_{\omega_X}^2 \right) \\ &\geq \frac{1}{\operatorname{tr}_{\omega_X} (f^* \omega_Y)} \left(\langle \operatorname{Ric}(\omega_X), f^* \omega_Y \rangle_{\omega_X} - g^{k\bar{l}} g^{i\bar{j}} (\partial_i f^\alpha) \overline{(\partial_{\bar{j}} f^\beta)} (\partial_k f^\gamma) \overline{(\partial_{\bar{l}} f^\delta)} S_{\alpha\bar{\beta}\gamma\bar{\delta}} \right) \\ &\geq Bu - A.\end{aligned}$$

In the second line from the bottom, we used the following inequality:

$$\begin{aligned}|\nabla \operatorname{tr}_{\omega_X} (f^* \omega_Y)|_{\omega_X}^2 &= g^{i\bar{j}} (\partial_i g^{k\bar{l}} h_{k\bar{l}}^*) (\partial_{\bar{j}} g^{p\bar{q}} h_{p\bar{q}}^*) = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} (\partial_i h_{k\bar{l}}^*) (\partial_{\bar{j}} h_{p\bar{q}}^*) \\ &= \sum_{i,k,p,\alpha,\beta} (\partial_i \partial_k f^\alpha) \overline{(\partial_{\bar{k}} f^\alpha)} (\partial_{\bar{i}} \partial_p f^\beta) (\partial_p f^\beta) \\ &\leq \sum_{k,p,\alpha,\beta} \left(|\partial_p f^\beta| |\partial_k f^\alpha| \left(\sum_i |\partial_i \partial_k f^\alpha|^2 \right)^{1/2} \left(\sum_j |\partial_j \partial_p f^\beta|^2 \right)^{1/2} \right) \\ &= \left(\sum_{k,\alpha} |\partial_k f^\beta| \left(\sum_i |\partial_i \partial_k f^\alpha|^2 \right)^{1/2} \right)^2 \\ &\leq \left(\sum_{l,\beta} |\partial_l f^\beta|^2 \right) \left(\sum_{i,k,\alpha} |\partial_i \partial_k f^\alpha|^2 \right) \\ &= \operatorname{tr}_{\omega_X} (f^* \omega_Y) g^{k\bar{l}} g^{i\bar{j}} (\partial_i \partial_k f^\alpha) \overline{(\partial_{\bar{j}} \partial_{\bar{l}} f^\beta)}.\end{aligned}$$

Here, we used the Cauchy-Schwarz inequalities. \square

The next proposition is the so-called ‘‘Jeffres’ trick’’.

Proposition 3.3 ([Jef00a, Section 4]). *Let X be a compact Kähler manifold, D be a smooth divisor, and β be a real number satisfying $0 < \beta < 1$. Let $s \in H^0(X, \mathcal{O}_X(D))$ be a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D , and h is a smooth Hermitian metric. Then, for any function $u \in C^{\alpha, \beta}$ and $\varepsilon > 0$, every maximum point of the function*

$$u_\varepsilon := u + \varepsilon |s|_h^{2\gamma}$$

on X belongs to $X \setminus D$ if $0 < 2\gamma < \alpha\beta$.

Proof. We assume that u_δ takes maximum at $x_0 \in D$. Let $(U, (z^1, \dots, z^n))$ be a holomorphic chart centered at x_0 satisfying $D \cap U = \{z^1 = 0\}$. By the definition of x_0 , for any $x = (z, 0, \dots, 0) \in U$, we have

$$\frac{|u(x) - u(x_0)|}{d_\beta(x, x_0)^\alpha} = \frac{|u(x) - u(x_0)|}{|z|^{\alpha\beta}} \geq \frac{\varepsilon |s|_h^{2\gamma}(x)}{|z|^{\alpha\beta}} \geq \frac{\varepsilon}{C} \frac{|z|^{2\gamma}}{|z|^{\alpha\beta}}.$$

Since $0 < 2\gamma < \alpha\beta$, the right hand side goes to ∞ as $z \rightarrow 0$. This contradicts with the definition of $C^{\alpha, \beta}$. \square

Theorem 1.4 and Theorem 1.5 can be shown in a similar manner. We only prove Theorem 1.4 here.

Proof of Theorem 1.4 (a). Since f can be represented as $(w^1, \dots, w^n) = ((z^1)^k, f_2(z), \dots, f_n(z))$ such that $D = \{z^1 = 0\}$ and $E = \{w^1 = 0\}$, the direct computation gives that f is locally Hölder continuous with respect to d_α and d_β if $\alpha \leq k\beta$. Combining with (3.2) and the definition of the cone metrics, $v := f^* \omega_Y^n / \omega_X^n$ is a $C^{\sigma, \beta}$ function for some $0 < \sigma < 1$. By Proposition 3.3, all maximum points of $v_\delta := v + \varepsilon |s|_h^{2\gamma}$ belong to $X \setminus D$ where γ is sufficiently small. Since v_ε is smooth on $X \setminus D$, we can apply the maximum principle argument to v_ε . The direct computation show that

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} |s|_h^{2\gamma} &= \sqrt{-1} \partial \bar{\partial} e^{\gamma \log |s|_h^2} = |s|_h^{2\gamma} (\gamma \sqrt{-1} \partial \bar{\partial} \log |s|_h^2 + \gamma^2 \sqrt{-1} \partial \log |s|_h^2 \wedge \bar{\partial} \log |s|_h^2) \\ &\geq -\gamma |s|_h^{2\gamma} \sqrt{-1} R_h. \end{aligned}$$

Therefore, there exists a constant $C > 0$ (which is independent of ε) satisfying

$$\Delta_{\omega_X} |s|_h^{2\gamma} \geq -C.$$

Let $x_0 \in X \setminus D$ be a maximum point of v_ε . At this point, by Proposition 3.1 (a), we have

$$0 \geq \Delta_{\omega_X} v_\varepsilon = \Delta_{\omega_X} v + \varepsilon \Delta_{\omega_X} |s|_h^{2\gamma} \geq v(nBv^{1/n} - A) - \varepsilon C.$$

Simple calculus show that the function $t \mapsto t^n(nBt - A) - \varepsilon C$ takes non-positive values exactly on some bounded interval $[0, T_\varepsilon]$ and $T_\varepsilon \rightarrow A/(nB)$ as $\varepsilon \rightarrow 0$. It follows that

$$v_\varepsilon(x_0) = v(x_0) + \varepsilon |s|_h^{2\gamma}(x_0) \leq T_\varepsilon^n + \varepsilon \sup_X |s|_h^{2\gamma}.$$

Since the right hand side does not depend on x_0 and x_0 is any maximum point of v_ε , this inequality holds on whole X . Therefore, we have the following inequality

$$v = v_\varepsilon - \varepsilon |s|_h^{2\gamma} \leq v_\varepsilon \leq T_\varepsilon^n + \varepsilon \sup_X |s|_h^{2\gamma}$$

on X . By taking $\varepsilon \rightarrow 0$, we obtain $v \leq (A/(nB))^n$. \square

Proof of Theorem 1.4 (b). By definition of the cone metric, we can easily see that for any $\varepsilon > 0$,

$$v_\varepsilon := |s|_h^{2(\ell+\varepsilon)} v = |s|_h^{2(\ell+\varepsilon)} \frac{f^* \omega_Y^n}{\omega_X^n}$$

tends to 0 as x approaches to D , where $\ell := \alpha - k\beta > 0$. Then, combining the Laplacian estimate in Proposition 3.1 (a), we have

$$\begin{aligned} \Delta_{\omega_X} \log v_\varepsilon &= -(\ell + \varepsilon) \operatorname{tr}_{\omega_X} (\sqrt{-1} R_h) + \Delta_{\omega_X} \log v \\ &\geq -(\ell + \varepsilon) C - A + nBv^{1/n}, \\ \Delta_{\omega_X} v_\varepsilon &\geq v_\varepsilon (-(\ell + \varepsilon) C - A + nBv^{1/n}). \end{aligned}$$

If $x_0 \in X$ is a maximum of v_ε , we can assume that $x_0 \in X \setminus D$. At this point, by applying the maximum principle, we have

$$v(x_0) \leq \left(\frac{A + (\ell + \varepsilon)C}{nB} \right)^n.$$

Therefore, we get

$$v_\varepsilon(x_0) \leq |s|_h^{\ell+\varepsilon}(x_0) \left(\frac{A + (\ell + \varepsilon)C}{nB} \right)^n \leq \left(\frac{A + (\ell + \varepsilon)C}{nB} \right)^n.$$

Since the right hand side does not depend on x_0 , this inequality holds on X . Taking $\varepsilon \rightarrow 0$, we obtain

$$|s|_h^{2\ell} \frac{f^* \omega_Y^n}{\omega_X^n} \leq \left(\frac{A + \ell C}{nB} \right)^n.$$

□

REFERENCES

- [Ahl38] L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), no. 3, 359–364, DOI: 10.2307/1990065, MR: 1501949.
- [CDS15a] X. X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197, DOI: 10.1090/S0894-0347-2014-00799-2, MR: 3264766.
- [CDS15b] X. X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π* , J. Amer. Math. Soc. **28** (2015), no. 1, 199–234, DOI: 10.1090/S0894-0347-2014-00800-6, MR: 3264767.
- [CDS15c] X. X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278, DOI: 10.1090/S0894-0347-2014-00801-8, MR: 3264768.
- [Che68] S.-S. Chern, *On holomorphic mappings of hermitian manifolds of the same dimension.*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), Amer. Math. Soc., Providence, R.I., 1968, pp. 157–170, MR: 0234397.
- [Don12] S. Donaldson, *Kähler metrics with cone singularities along a divisor*, Essays in mathematics and its applications, Springer, Heidelberg, 2012, pp. 49–79, DOI: 10.1007/978-3-642-28821-0_4, MR: 2975584.
- [Jef00a] T. D. Jeffres, *Schwarz lemma for Kähler cone metrics*, Internat. Math. Res. Notices (2000), no. 7, 371–382, DOI: 10.1155/S1073792800000210, MR: 1749739.
- [Jef00b] T. D. Jeffres, *Uniqueness of Kähler-Einstein cone metrics*, Publ. Mat. **44** (2000), no. 2, 437–448, DOI: 10.5565/PUBLMAT.44200.04, MR: 1800816.

- [Lu68] Y.-C. Lu, *Holomorphic mappings of complex manifolds*, J. Differential Geometry **2** (1968), 299–312, MR: 0250243.
- [Tia15] G. Tian, *Kähler-Einstein metrics on Fano manifolds*, Jpn. J. Math. **10** (2015), no. 1, 1–41, DOI: 10.1007/s11537-014-1387-3, MR: 3320994.
- [Yau78] S.-T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197–203, MR: 0486659.

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